

IND- AND PRO- DEFINABLE SETS

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ABSTRACT. We describe the ind- and pro- categories of the category of definable sets, in some first order theory, in terms of points in a sufficiently saturated model.

1. INTRODUCTION

Given the direct limit Y of some system Y_i in a given category, the morphisms from Y to another object X are described, by definition, as certain collections of morphisms from each Y_i to X . In contrast, there is, in general, no simple description of morphisms in the other direction, from X to Y . However, if the category in question is, for example, a category of topological spaces, and X is compact, then any morphism from X to Y will factor via some Y_i .

The category $\text{Ind}(\mathcal{C})$ of ind-objects of a category \mathcal{C} is a category containing the original category \mathcal{C} , in which any filtering system has a limit, and the objects of the original category are “compact” in the above sense. This construction, which appears in [1], can be applied to any category, and is described below. The dual construction, of the category of pro-objects, is described as well.

In the context of first order logic, and definable sets, there is a natural notion of compactness, and given a system of definable sets, one may compute limits of their points in a given model. The purpose of this note is to describe how the categorical notions of ind- and pro- objects apply to definable sets, and in particular to describe the categories of ind- and pro- definable sets in terms of points in a model. The main results are proposition 4, which explains how to compute the M points of P , where M is any model and P is an ind-definable (or pro-definable) set, and proposition 7, which describes morphisms in terms of such points. The final statement of the results is in corollary 8.

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2. CATEGORICAL NOTIONS

We begin by recalling some general notions from category theory. The reference to all this is [1]. Let \mathcal{C} be a category (which we assume to be small), $\hat{\mathcal{C}}$ the category of presheaves on \mathcal{C} (i.e., contra-variant functors from \mathcal{C} to the category of sets), and $\mathbf{y} : \mathcal{C} \rightarrow \hat{\mathcal{C}}$ the Yoneda embedding, given by $\mathbf{y}(X)(Z) = \text{Hom}(Z, X)$.

A *filtering category* is a small category I such that:

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- For any two objects i, j of I , there are morphisms $i \rightarrow k$ and $j \rightarrow k$ for some object k .
- For any two morphism $t_1, t_2 : i \rightarrow j$ there is a morphism $s : j \rightarrow k$ with $s \circ t_1 = s \circ t_2$.

A *filtering system* in \mathcal{C} is a functor from a filtering category to \mathcal{C} . Such a system will be denoted (X_i) , where X_i is the object of \mathcal{C} associated with i . We now define $\text{Ind}((X_i))$, the *ind-object* of \mathcal{C} associated with the system (X_i) , to be $\varinjlim \mathbf{y}(X_i)$ (an object of $\widehat{\mathcal{C}}$.) Recall that direct limits in $\widehat{\mathcal{C}}$ can be computed “pointwise”. Thus, we have for any object Y of \mathcal{C} ,

$$\text{Hom}(Y, \text{Ind}((X_i))) = \text{Ind}((X_i))(Y) = \varinjlim \text{Hom}(Y, X_i)$$

The category $\text{Ind}(\mathcal{C})$ is defined to be the full subcategory of $\widehat{\mathcal{C}}$ of presheaves isomorphic to $\text{Ind}((X_i))$ for some filtering system (X_i) .

Any directed partially ordered set can be viewed as a filtered category, and conceptually a filtering system can be thought of as a partially ordered one. In fact, it can be shown that any filtering system is isomorphic to a partially ordered one. However, in some cases (such as the proof of proposition 4 below), the natural index category has the more general form.

The category of pro-objects $\text{Pro}(\mathcal{C})$ is defined by dualising: it is defined to be $\text{Ind}(\mathcal{C}^\circ)^\circ$, where \mathcal{C}° denotes the opposite category to \mathcal{C} . We describe it explicitly in terms of \mathcal{C} itself: let $\check{\mathcal{C}} = \widehat{\mathcal{C}^\circ}$ be the category of co-variant functors from \mathcal{C} to sets, $\check{\mathbf{y}} : \mathcal{C} \rightarrow \check{\mathcal{C}}$ the (contra-variant) Yoneda embedding. Given a co-filtering system (X_i) in \mathcal{C} (i.e., a contra-variant functor from a filtering category to \mathcal{C}), the associated pro-object is defined to be the functor $\text{Pro}((X_i)) = \varprojlim \check{\mathbf{y}}(X_i)$. For any object Y of \mathcal{C} we get

$$\begin{aligned} \text{Hom}_{\text{Pro}(\mathcal{C})}(\text{Pro}((X_i)), Y) &= \text{Hom}_{\check{\mathcal{C}}}(Y, \text{Pro}((X_i))) = \\ &= \text{Pro}((X_i))(Y) = \varprojlim \text{Hom}_{\mathcal{C}}(X_i, Y) \end{aligned}$$

More generally, we have the following formulas for the morphism sets in the *Pro* and *Ind* categories:

$$(1a) \quad \text{Hom}(\text{Ind}((X_i)), \text{Ind}((Y_j))) = \varinjlim_i \varinjlim_j \text{Hom}(X_i, Y_j)$$

$$(1b) \quad \text{Hom}(\text{Pro}((X_i)), \text{Pro}((Y_j))) = \varprojlim_j \varprojlim_i \text{Hom}(X_i, Y_j)$$

It follows that any presheaf P on \mathcal{C} extends canonically to $\text{Pro}(\mathcal{C})$ by setting $P(\text{Pro}((X_i))) = \varinjlim P(X_i)$: a map of pro-objects

$$f : \text{Pro}((X_i)) \rightarrow \text{Pro}((Y_j))$$

is represented by a sequence of maps $f_j : X_{i_j} \rightarrow Y_j$, hence we get maps $P(f_j) : P(Y_j) \rightarrow P(X_{i_j})$ that represent a map from $\varinjlim P(Y_j)$ to $\varinjlim P(X_i)$. Likewise, any functor from \mathcal{C} to sets can be extended to a functor on $\text{Ind}(\mathcal{C})$.

Given an object X of \mathcal{C} , the category \mathcal{C}/X is defined to have \mathcal{C} -morphisms $Y \rightarrow X$ as objects, and \mathcal{C} -morphisms over X as morphisms. Then $\text{Ind}(\mathcal{C}/X) = \text{Ind}(\mathcal{C})/X$ and $\text{Pro}(\mathcal{C}/X) = \text{Pro}(\mathcal{C})/X$. The first assertion follows by definition (and is true

when X is replaced by any presheaf), while the second uses the fact that the systems are filtered.

We are going to use the following lemma, which describes a sufficient condition for a morphism with a section to be an isomorphism:

Lemma 1. *Let $f : \text{Ind}((X_i)) \rightarrow Y$, $g : Y \rightarrow X_0$ be two morphisms, such that $f_0 \circ g$ is the identity on Y . Assume that for any i , there is a morphism $t_i : X_i \rightarrow X_j$ in the system, such that for any two morphisms $h_1, h_2 : V \rightarrow X_i$, if $f_i \circ h_1 = f_i \circ h_2$, then $t_i \circ h_1 = t_i \circ h_2$ (this is the formal analogue of saying that f_j is injective on the image of t_i .)*

Then f is an isomorphism with inverse g .

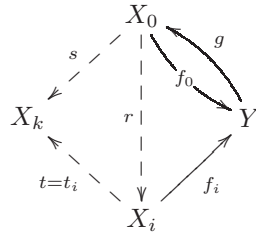
Proof. First note that for any filtering system (X_i) and an object X in the system, the (full) subsystem consisting of all objects that have a system morphism from X is isomorphic (in the *Ind* category) to the original one. Thus we may assume that there is a system morphism from X_0 to any other object in the system.

To show that g is the inverse of f , we need to show that $g \circ f$ is the identity on $\text{Ind}((X_i))$ (the other composition is the identity by assumption.) This amounts to showing that for any i , $g \circ f_i$ is identified with some morphism in the system (X_i) . In other words, we need to show that there are morphisms $t : X_i \rightarrow X_k$, $s : X_0 \rightarrow X_k$ such that $s \circ g \circ f_i = t$ (In fact, for any object Z ,

$$\begin{aligned} (g \circ f)_Z(\text{Ind}((X_i))(Z)) &= (g \circ f)_Z(\lim_{\substack{\longrightarrow \\ i}} \text{Hom}(Z, X_i)) = \\ &= \lim_{\substack{\longrightarrow \\ i}} \{g \circ f_i \circ u \mid u \in \text{Hom}(Z, X_i)\} \end{aligned}$$

If the above condition holds, the map taking $u : Z \rightarrow X_i$ to $g \circ f_i \circ u$ is an isomorphism of the limit sets, since $s \circ g \circ f_i \circ u = t \circ u$.)

The situation is this:



We should find X_k , t and s , such that the external square commutes. We take $t = t_i$, as promised by the assumption. By the reduction above, there is some morphism r from X_0 to X_i . We set $s = t \circ r$. Thus we should prove that $t \circ r \circ g \circ f_i = t$. By the property of t , it is enough to show that $f_i \circ r \circ g \circ f_i = f_i$. But this is true since $f_i \circ r \circ g = f_0 \circ g = 1_Y$. \square

Remark 2. In the case that \mathcal{C} has finite inverse limits, we may replace the arbitrary V by $X_i \times_Y X_i$ (and the h_i by the projections.) Thus, in this case we get the following simpler condition:

Let \mathcal{C} be a category with finite inverse limits. Let $f : \text{Ind}((X_i)) \rightarrow Y$, $g : Y \rightarrow X_0$ be morphisms, such that $f_0 \circ g$ is the identity on Y . Assume that for any i , there is a morphism $t_i : X_i \rightarrow X_j$ in the system, such that the map $X_i \times_{X_j} X_i \rightarrow X_i \times_Y X_i$ is an isomorphism.

Then f is an isomorphism with inverse g .

Remark 3. For convenience, we rephrase the above statement in terms of *Pro* objects:

Let \mathcal{C} be a category with finite direct limits. Let $f : Y \rightarrow \text{Pro}((X_i))$, $g : X_0 \rightarrow Y$ be morphisms, such that $g \circ f_0$ is the identity on Y . Assume that for any i , there is a morphism $t_i : X_j \rightarrow X_i$ in the system, such that the map $X_i \amalg_Y X_i \rightarrow X_i \amalg_{X_j} X_i$ is an isomorphism.

Then f is an isomorphism with inverse g .

3. THE CASE OF DEFINABLE SETS

We now consider the model theoretic setting. The basic terminology is explained, for example, in [3]. Let T be a first order theory, \mathcal{M} the opposite category to the category of models of T and elementary maps, and \mathcal{D} the category of definable sets and definable functions between them (the word “definable” will mean definable over 0.) The relationship between them is described by the faithful functor $p : \mathcal{D} \rightarrow \widehat{\mathcal{M}}$, given by $p(X)(M) = X(M)$. We first show that this functor has a natural extension to the whole category $\widehat{\mathcal{D}}$ of presheaves on \mathcal{D} .

Proposition 4. *There is a fully faithful functor $d : \mathcal{M} \rightarrow \text{Pro}(\mathcal{D})$ such that for any definable set X and model M , $\text{Hom}(d(M), X) = X(M)$. In particular, for any presheaf or functor F on \mathcal{D} , $F(M)$ is well defined.*

Before giving the proof, we roughly explain the idea. A basic property of any definable set X is that if $a \in X(M) \subseteq M^n$, then the whole type of a (over 0) is contained in X , and we would like this property to hold for an arbitrary presheaf. Since a type is just an example of a pro-definable set, $X(M)$ can be written as

$$\begin{aligned} X(M) &= \coprod_{a \in M^n} \text{Hom}(tp(a), X) = \coprod_{a \in M^n} \varinjlim_{\substack{Y \text{ with} \\ a \in Y(M)}} \text{Hom}(Y, X) = \\ &= \varinjlim_{(Y, a \in Y(M))} \text{Hom}(Y, X) \end{aligned}$$

where $\text{Hom}(X, Y)$ here is taken in the sense of inclusions (so $\text{Hom}(X, Y)$ contains one element if $X \subseteq Y$, and is empty otherwise.) When we wish to describe this observation in terms of the pro-definable category, we run into several problems: first, we obtain distinct systems for distinct values of n . Second, these systems are not co-filtering. Finally, it is not clear how to distinguish inclusions inside the category. Fortunately, all of these problems are solved by replacing inclusions by arbitrary definable maps, as we do in the proof, below.

Proof of proposition 4. Given a model M , let $(X_{(X,a)})$ be the system where $a \in M$, X is a definable set with $a \in X(M)$, and $X_{(X,a)} = X$ (since we no longer distinguish inclusions, we also don’t distinguish between elements and tuples.) The morphisms from $X_{(X,a)}$ to $Y_{(Y,b)}$ are definable maps $f : X \rightarrow Y$ with $f(a) = b$. This system is cofiltering since all finite inverse limits exist in \mathcal{D} . We abbreviate $X_{(X,a)}$ as X_a and set $d(M) = \text{Pro}((X_a))$. We first show that for any definable set Y , we have a canonical bijection $\text{Hom}(d(M), Y) \rightarrow Y(M)$. Indeed, by definition

$$\text{Hom}(d(M), Y) = \varinjlim \text{Hom}(X_a, Y)$$

So to give a map from $\text{Hom}(d(M), Y)$ to $Y(M)$ is the same as to give a matching collection of maps from each $\text{Hom}(X_a, Y)$ to $Y(M)$. For each $f \in \text{Hom}(X_a, Y)$ we

assign $f(a)$. To show that this map is a bijection, we note that the map in the other direction is given by assigning to each $a \in Y(M)$ the identity map on $Y = Y_a$. This is, in fact, the inverse, since any definable map $f : X_a \rightarrow Y$ is identified with the identity map when “restricted” to the graph of f . More verbosely, let $f : X_a \rightarrow Y$ represent an element in $\text{Hom}(d(M), Y)$. Applying the composition of the two maps, we get the identity map on $Y_{f(a)}$. If Γ is the graph of f , the two projections give maps in the system $\Gamma_{(a, f(a))} \rightarrow X_a$ and $\Gamma_{(a, f(a))} \rightarrow Y_{f(a)}$ that identify f and the identity on $Y_{f(a)}$.

To define d on morphisms, we first note that, by what was just shown, given two models M and N ,

$$\text{Hom}(d(M), d(N)) = \varinjlim \text{Hom}(d(M), X_a) = \varinjlim X_a(M)$$

(where the limit is taken over pairs (X, a) with $a \in N$.) Thus, to define the map $d : \text{Hom}_{\mathcal{M}}(M, N) \rightarrow \text{Hom}(d(M), d(N))$ we need to assign, to each elementary map $f : N \rightarrow M$ a compatible system of points in the $X_a(M)$. We do it by taking the point $f(a)$. In the other direction, given a matching collection of points, we construct a map from N to M by assigning to a point $a \in N$ the point specified for $U_a(M)$ (where U is the universe, $x = x$.) To show that this map is elementary, we note that if, for some definable set $X \subseteq U^n$, we have $\bar{a} \in X(N)$, $f(\bar{a})$ is the specified point in $X_{\bar{a}}(M)$: $f(a_1, \dots, a_n) = (f(a_1), \dots, f(a_n))$, and each $f(a_i)$ is the specified point for U_{a_i} . The projections from $U^n_{\bar{a}}$ to the U_{a_i} now show that $f(\bar{a})$ is the specified point for $U^n_{\bar{a}}$, hence, because of the inclusion $X \subseteq U^n$, for $X_{\bar{a}}(M)$.

This concludes the construction of the embedding. The last remark follows directly from the remarks above. Explicitly, for P a presheaf on \mathcal{D} , and F a functor from \mathcal{D} to sets, we have for any model M :

$$(2a) \quad F(M) = \text{Hom}_{\mathcal{D}}(F, d(M)) = \text{Hom}_{\mathcal{D}}(F, \varinjlim \check{Y}(X_a))$$

$$(2b) \quad P(M) = \varinjlim P(X_a)$$

□

Remark 5. Instead of viewing definable sets as functors on the category of models, we may, conversely, view a model as a functor on the definable sets. From this point of view, the construction of $d(M)$ (for a general functor M) is mentioned in [2] as the *Grothendieck construction*. Unfortunately, I do not know the purpose of this construction in general.

We are interested in two special cases of the formulas (2): let $P = \text{Ind}((Z_i))$, $F = \text{Pro}((Y_i))$. In this case we obtain:

$$\begin{aligned} \text{Ind}((Z_i))(M) &= \varinjlim \text{Ind}((Z_i))(X_a) = \varinjlim_{(X,a)} \varinjlim_i \text{Hom}(X_a, Z_i) = \\ &= \varinjlim_i \varinjlim_{(X,a)} \text{Hom}(X_a, Z_i) = \varinjlim_i Z_i(M) \\ \text{Pro}((Y_i))(M) &= \text{Hom}_{\mathcal{D}}(\text{Pro}((Y_i)), d(M)) = \\ &= \varprojlim_i \text{Hom}_{\mathcal{D}}(Y_i, d(M)) = \varprojlim_i Y_i(M) \end{aligned}$$

Thus, to compute the points of a pro-definable set in a model M , we need to choose a presentation of it as system, and compute the inverse limit of the associated system of sets (and similarly for ind-definable sets.)

We may now identify these sets of points with some familiar model theoretic objects. Let \mathfrak{p} be any partial type. The definable sets comprising it form a co-filtering system, with all maps the inclusions. The last equations says that computing the M points of \mathfrak{p} , viewed as pro-definable set, coincides with computing its M points as a type, i.e., taking the intersection of the M points of the definable sets in \mathfrak{p} . The fact that two such system that give the same pro-definable set also give the same set of points means that this set of points is determined by the set of definable sets containing \mathfrak{p} .

A partial type such as above is always contained in some definable set. There is a more general construction, called a $*$ -type, that consists of the intersection of formulas in an arbitrary set of variables. Such types are similarly examples of pro-definable sets.

Analogously, an increasing union of definable set is an example of an ind-definable set. A more complicated example can be formulated as follows: let E_i be definable equivalence relations on a definable set X , indexed by natural numbers i , such that for $i > j$, E_i is coarser than E_j . Let E be the equivalence relation saying that xEy if xE_iy for some i . Then E is the union of the E_i and thus an example of an ind-definable equivalence relation. The quotient of X by E is another example of an ind-definable set.

Our next purpose is to describe the morphisms between the new objects in terms of their points in models. Considering equations (1) again, we see in particular that any morphism from $\text{Ind}((X_i))$ to $\text{Ind}((Y_j))$ gives rise to a filtering system Γ_i of the corresponding graphs of functions from X_i to Y_{j_i} . Similarly, a morphism of pro-definable sets gives rise to a cofiltering system. Each such system is isomorphic to its domain X_i , and therefore induces a function on the level of points from $\text{Ind}((X_i))(M)$ to $\text{Ind}((Y_j))(M)$ (and similarly for pro-definable sets.) We would like to show that conversely, any ind-definable set that gives rise to a function on the points of every model (equivalently, saturated enough model) induces a morphism.

We first restate the compactness theorem in this language:

Proposition 6. *Let κ be a cardinal bigger than the cardinality of the index category (i.e., the cardinality of the disjoint union of the morphism sets.)*

- (1) *Let $f : \text{Ind}(X_i) \rightarrow Y$ be a morphism such that for some κ -saturated model M , $f_M : \text{Ind}(X_i)(M) \rightarrow Y(M)$ is a bijection. Then f is an isomorphism.*
- (2) *Let $f : Y \rightarrow \text{Pro}(X_i)$ be a morphism such that for some κ -saturated model M , $f_M : Y(M) \rightarrow \text{Pro}(X_i)(M)$ is a bijection. Then f is an isomorphism.*

Proof. In each case, let f_i be the maps corresponding to the morphism f . Note that for definable sets and maps, the claims are true by definition (f is an isomorphism in this case.) We shall use the criterion of remark 2 (and remark 3.)

- (1) We will find g and t_i as required by remark 2. We first show that for some k , f_k is onto. In fact, the collection of sets $f_i(X_i)(M)$ is a small covering of $Y(M)$, hence it has a finite sub-cover. Since the system is filtering, there is an X_k above all the sets in the sub-cover.

We next note that the t_i condition requires, in this case, for each i , a definable map $t_i : X_i \rightarrow X_j$ in the system such that $f_i(x) = f_i(y)$ defines

the same set as $t_i(x) = t_i(y)$. This again holds by compactness: consider the set of formulas consisting of the formula $f_i(x) = f_i(y)$, and for each morphism $t : X_i \rightarrow X_j$ in (X_i) , $t(x) \neq t(y)$. This set expresses the fact that the elements $x, y \in X_i$ determine distinct elements of $\text{Ind}((X_i))(M)$, that have the same image under f . Therefore it is not satisfied in M . Since this collection is small, a finite subset is not satisfied. Therefore, there is some $t_i : X_i \rightarrow X_j$ such that $f_i(x) = f_i(y)$ implies $t_i(x) = t_i(y)$.

In particular, this means that f_j is injective on the image of t_i . Let X_0 be the codomain of t_k (for the X_k found above.) Then f_0 restricted to the image of t_k is a bijection. We take g to be the inverse of this restriction.

- (2) The proof is dual, using remark 3. The only complication here is that the category of definable sets does not, in general, have finite direct limits. The assumption that such limits exist is called *elimination of imaginaries (EI)*. However, for the specific purpose of the condition in remark 3, we do not actually need these limits. In our case, the condition simply translates to saying that f_i and t_i have the same image. Such t_i can be obtained by compactness, using the surjectivity of the limit map, as in the dual case.

Further, by considering the formulas $f_i(x) \neq f_i(y)$, we see that there is an X_k such that f_k is injective. In particular, we have $t_k : X_0 \rightarrow X_k$, such that f_k is a bijection between Y and the image of t_k . Taking $g = f_k^{-1} \circ t_k$, all the conditions of the lemma are satisfied.

□

The promised description of morphisms is just the extension of this criterion to the entire category:

Proposition 7. *Let κ be a cardinal bigger than the cardinality of the index category, M a κ saturated model. Let X and Y be ind- (or pro-) definable sets, $f : X \rightarrow Y$ a morphism that induces a bijection on M points. Then f is an isomorphism.*

In particular, there is a natural bijection between $\text{Hom}(X, Y)$ and sub-objects of $X \times Y$ whose set of M -points is a function from $X(M)$ to $Y(M)$.

Proof. We prove for the *Ind* category, the *Pro* case is dual. We have $f : \text{Ind}(X_i) \rightarrow \text{Ind}(Y_i)$. We first note that for any map $f : P \rightarrow \text{Ind}(Y_i)$ where P is a presheaf, f is an isomorphism if and only if for all j , the pullback $f_j : P \times_{\text{Ind}(Y_i)} Y_j \rightarrow Y_j$ is an isomorphism. Indeed, given inverses g_j to the f_j , their composition with the projection to P forms a matching family of maps from the Y_i to P , and therefore yields a map from $\text{Ind}(Y_i)$ to P , inverse to f .

Furthermore, if P itself is ind-definable, $P = \text{Ind}(X_i)$, we have

$$P \times_{\text{Ind}(Y_i)} Y_j = \text{Ind}(X_i \times_{\text{Ind}(Y_i)} Y_j) = \text{Ind}(X_i \times_{Y_{k_i}} Y_j)$$

On the other hand, since taking M points is represented by a pro-definable set, it preserves pullbacks. Therefore, if f_M is a bijection of M points, so is f_{j_M} , for any j . By proposition 6, f_j is an isomorphism.

The description of the morphism sets is the interpretation of this statement for the projection map from a sub object R of $X \times Y$ to X .

□

We may summarise the results of this section as follows:

Corollary 8. *Let M be a κ -saturated model.*

The functor of “taking M points” is an equivalence of categories between the category $\text{Pro}_\kappa(\mathcal{D})$ of pro-definable sets representable by systems of length less than κ , and the sub-category of the category of sets whose objects and morphisms are inverse co-filtered limits of M points of definable sets, of length less than κ .

Similarly, the same functor is an equivalence of categories between the category $\text{Ind}_\kappa(\mathcal{D})$ of ind-definable sets representable by systems of length less than κ , and the sub-category of the category of sets whose objects and morphisms are direct filtered limits of M points of definable sets, of length less than κ .

Finally, we note that definable sets are given with canonical inclusions (in the “universe”.) For example, in our terminology, any two points are identified. If we wish to remember the inclusion of the definable sets in some definable set X , we work in the category \mathcal{D}/X , and all results continue to hold. This way we get pro-definable subsets of X . These sets are called also ω -definable.

REFERENCES

1. A. Grothendieck et al., *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos*, Springer-Verlag, Berlin, 1972, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat, Lecture Notes in Mathematics, Vol. 269. MR MR0354652 (50 #7130) [1](#), [2](#)
2. Saunders Mac Lane and Ieke Moerdijk, *Sheaves in geometry and logic*, Universitext, Springer-Verlag, New York, 1994. MR **MR1300636** (96c:03119) [5](#)
3. Gerald E. Sacks, *Saturated model theory*, W. A. Benjamin, Inc., Reading, Mass., 1972. MR MR0398817 (53 #2668) [3](#)

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